**CALCULUS – KEY TERMS & MAIN RESULTS**

You are required to achieve these knowledge and skills:

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| 1. Chapter 1    1. Find the limit of a function using graph and algebraic methods. |
| * 1. Identify discontinuous points of a function. |
| 1. Chapter 2    1. Calculate rate of change and simply a difference quotient. |
| * 1. Find derivative as the limit of a difference quotient, and recognize derivative as slope of tangent line   and as instantaneous rate of change. |
| * 1. Find derivatives using basic formulas, the product and quotient rules. |
| * 1. Compute the composition of two functions; find derivatives using the chain rule.   2. Find derivatives using implicit differentiation, and use it to solve related rates problems. |
| 1. Chapter 3    1. Find critical numbers and compute absolute extrema of a function. |
| * 1. Determine local extrema of a function; describe where the function is increasing and decreasing. |
| * 1. Identify inflection points of a function and describe its concavity. |
| * 1. Use derivatives to solve applications involving rate of changes. |
| * 1. Use derivatives to solve optimization problems. |
| * 1. Find differential and marginal values of a function. |
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| 1. Chapter 5    1. Use Riemann sum to approximate the area; recognize definite integral as the limit of Riemann sums. |
| * 1. Recognize antiderivatives as indefinite integrals; find integrals using basic formulas   and the methods of substitution and integration by parts. |
| * 1. Use properties of integrals to manipulate integrals; find the area between two curves. |
| * 1. Use integration to solve problems in other sciences.   2. Find the equilibrium point of demand and supply functions; compute producer and consumer surplus.   3. Use growth and decay models to find present and future value of an investment,   and the accumulated present and future value of a continuous income stream.   * 1. Determine if an improper integral is convergent; find the present value of a perpetual   continuous money flow and other related applications. |
| 1. Chapter 6. |
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| * 1. Check if a function satisfies a differential equations; solve basic   and separable differential equations. |
| * 1. Compute values and find partial derivatives of a function of multi-variables;   recognize marginal productivities as partial derivatives. |
| * 1. Identify local extrema of a multi-variable function. |
| * 1. Solve optimization problems involving functions of many variables. |
| * 1. Find the best-fit line of data pairs, and use it to predict future values. |
| * 1. Use Lagrange technique to find maximum/minimum values of a   multi-variable function with constraints. |

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| **Key terms** | **Problems with solutions** | **Exercises - Do yourself** |
| **Chapter 1. Differentiation** | | |
| Find the limit of a function | ***E1x.*** Find (if any)  ***Solution.***   * means x is near 3 and x < 3 🡺 |x – 3| =   -(x-3) 🡺   * : |x – 3| = x-3 and * ≠   🡺  [Trick: try with x near 3, for example, x = 3.01, x = 2.99 and consider the results.]  ***Ex2.*** Use the graph of function k to answer each of the following.    a/ Find  b/ Find  Solution.  a/  b/ | **1/** Find the limit  a/  b/  c/  2/ Find (if any)  **3/** Find |
| Continuity, continuous,  test for **continuity**  **(**at x = a) | ***Ex1.*** Given the function.  Find m such that f is **continuous** at x = 3.  Solution.   * f(3) = 6 * f(x) = 32 – 3 = 6 * f(x) = 3 – m * f is **continuous** at x = 3 limx→3f(x) = f(3)   f(x) = f(x) = f(3) 3 – m = 6 m = -3  ***Ex2.*** For the function given by the graph below, answer each of the following:  a/ What is g(-2)?  b/ What is ?  c/ What is ?  d/ What is ?  e/ Is g continuous over (-6, 6)?    ***Solution.***  a/  b/  c/  does not exist  d/  does not exist 🡺g is not continuous at x = -2 🡺 not continuous over (-6, 6). | **4/**  a/ Find m such that the function    is ***continuous*** at x = 2.  b/ Find m such that the function    is ***continuous*** at x = 2.  **5/** Find all values of a, b such that  is **continuous** at x = 1 and x = 2.  **6/** For the function given by the graph below, answer each of the following:  a/ What is g(1)?  b/ What is ?  c/ What is ?  d/ What is ?  e/ Is g continuous over (-6, 6)? |
| Derivative, y’, f’(x), dy/dx | ***Ex.*** Find the ***derivative*** of the function with respect to the indicated variable.  a/  b/  c/  d/  ***Solution.***  a/  b/  c/  d/ | **7/** Find the ***derivative*** of the function with respect to the indicated variable.  a/  b/  c/  d/ e/  f/ Find |
| Difference quotient    rate of change (rate)  and derivative | ***Ex1.*** Find  ***Solution.***   * Recall that * Let f(x) = x3, f(3) = 33 = 27 and f’(x) = 3x2 * = 3.(32) = 27.   ***Ex2.*** Find the ***difference quotient*** of the function f(x) = x2.  ***Solution.***   * f(a+h) = (a+h)2 = a2 + 2ah + h2 * f(a) = a2 * difference quotient:   =  (because h ≠0)  ***Ex3.*** The population of a city grows from an initial size of 5000 to a size P, given by P(t)= 5000 + 20t2, where t is in years. Find the growth ***rate***.  ***Solution.***  Growth rate = P’(t) = 40t. | **8/** Find  **9/** Find the ***difference quotient*** of the function f(x) = 1/x.  **10/** For f(x)= x2, find the ***difference quotient*** when: a = 3 and h = 0.1  **11/** Suppose the temperature T of a person during an illness is given by T(t) = -t2+2t+15, where T is the temperature, in degree Fahrenheit, at time t, in days. Find the ***rate of change*** of the temperature with respect to time.  (Hint: Find T’(t))  **12/** Given f(x) = x2 - x. Find the ***average rate of change*** with x1 = 2 and x2 = 4.  (Hint: evaluate  ) |
| **Slope, Tangent line**  y= f(x) at a:  y = f’(a)(x-a) + f(a) | ***Ex.*** Find an equation of the **tangent line** to the curve at the point (1, 2).  ***Solution.***   * // ***slope*** of the tangent line | **13/** Given the curve y = x3 – 2x.  a/ Find the **tangent line** of the curve at the point (2, 4).  b/ Find the point on the graph of the curve at which the ***tangent line*** has ***slope*** 1. |
| (gf)’(x) = g’(f(x)).f’(x)  = g’(u).u’(x) | ***Ex1.*** Given f(u) = , g(x) = u = 1 + 3x2,  find (fg)’(1).  ***Solution.***   * Let u = g(x), then u’(x) = 6x * f’(u) = = * (fg)’(x) = f’(g(x)).g’(x) = f’(u).u’(x)   = 6x  (fg)’(1) = 6.  ***Ex2.*** Suppose H(x) = (x – 1)3 can be expressed as (fg)(x), and g(x) = x – 1, what is f(x)?  ***Solution.***  We have, (fg)(x) = f(g(x)) = f(x-1) = H(x) = (x-1)3  So, if t = x – 1, we have f(t) = t3.  Conclusion: f(x) = x3.  ***Ex3.*** Given f(u) =, find .  Solution.  Let u = 1/x.  Based on the chain rule,  = f’(u).u’(x) = .= . | **14/**  a/ Given f(u) = u2, g(x) = 1 + 2x.  Find (fg)’(2).  b/ Given f(u) = u2, find .  **15/** Given F(x) = f(g(x)), and f(-2) = 8, f’(-2) = 4, f’(5) = 3, g(5) = -2, g’(5) = 6. Find F’(5).  **16/** Suppose H(x) = x3 – 5 can be expressed as (fg)(x), and f(x) = x – 5, what is g(x)? |
| find **dy/dt** (rate of y) when given **dx/dt** (rate of x), x and y. | ***Ex1***. Given x2 + y3 = 12 and dx/dt = -3, find dy/dt when x = 2.  ***Solution.***  x2 + y3 = 12 🡺 y = 2 if x = 2   * (x2 + y3) = (12) * 2x + 3y2 = 0 * 2.2.(-3) + 3.(2). = 0 * = 2.   ***Ex2.*** Find the ***rate of change*** of total revenue with respect to time. Assume that R(x) is in dollars and  R(x) = 12x - x2; when x = 5 and dx/dt = 10 units per day.  ***Solution.***  The ***rate of change*** of total revenue with respect to time is  Based on the *chain rule*, = R’(x).x’(t) = (12 – 2x)(10) = (12 - 2·5)(10) = 20 (USD/day).  ***Ex3.*** Suppose that the price p, in dollars, and number of sales, x, find the ***rate*** at which the total revenue R = xp is changing when x = 5, p = 12 and dp/dt = 1.5; dx/dt = 2.  ***Solution.***  R = xp 🡺 R’(t) = x’(t).p + x.p’(t)  Or  = 2.12 + 5(1.5) = 24 + 7.5 = 31.5 | **17/**  a/ Given x3 + y3 = 9 and dx/dt = -3, find dy/dt when x = 2.  b/ If dx/dt = 3, and x = p3 – 1/p, find dp/dt.  **18/**  a/ Two variable quantities A and B are found to be related by the equation A3 + B3 = 9. What is the ***rate of change*** dA/dt at the moment when A= 1 and dB/dt= 5?  b/ Find the ***rate of change*** of total cost with respect to time. Assume that C(x) is in dollars: C(x) = 7x2 + 4 when x= 10 and dx/dt  = 15 units per day.  c/ The volume of a cantaloupe is given by V = (4/3)πr3. The radius, r, is growing at the ***rate*** of 0.5 cm/week, at a time when the radius is 6.5 cm. ***How fast*** is the volume changing at that moment? |
| Find dy/dx by **implicit differentiation**. | ***Ex.*** Use **implicit differentiation** to ﬁnd an equation of the **tangent line** to the curve x2 + xy + y2 = 3 at the given point (1, 1).  ***Solution.***  x2 + xy + y2 ) = 3)   * (x2)’ + (xy)’ + (y2)’ = 0 * 2x + x’y + xy’ + 2y.y’ = 0 * 2x + y + (x+2y).y’ = 0 * At the point (x = 1, y = 1): y’(1) = -3/3 = -1   Equation of the **tangent line**:  y = y’(1)(x – x0) + f(x0)   * y = -(x – 1) + 1 * y = -x + 2. | **19/** Find dy/dx by **implicit differentiation**.  x2 + xy - y2 + x = 2.  **20/ *Differentiate implicitly*** to find dy/dx, where x2 – y2 = 1.  **21/** Use **implicit differentiation** to ﬁnd an equation of the **tangent line** to the curve x2 +2xy - y2 + x = 2 at the (1, 2). |
| dy and Δy  dy = f’(x)dx  Δy ≈ f’(x)dx | ***Ex1.*** Find Δy and y'.Δx.  (Round the result to two decimal places, respectively for y= x3, x = 2, and Δx = 0.05).  ***Solution.***  Δy = y(2 + Δx) – y(2) = (2.05)3 – 23 = 0.615125 ≈ 0.62  y'.Δx = (3x2).Δx = 3\*(22)\*(0.05) = 0.6.  Note that Δy ≈ y’Δx.  ***Ex2.*** For y = x3- 2x, find dy when x = 3 and dx = 0.01.  ***Solution.***  dy = y’dx = (3x2 – 2)dx = (3.32 – 2)\*0.01 = 0.25. | **22/** Find Δy and y'.Δx for y = x4, x = 2, and Δx = 0.01.  Round to two decimal places.  **23/** For y = (x-1)3, find dy when x = 2 and dx = 0.01 |
| **Chapter 2. Applications of differentiation** | | |
| **critical numbers** | ***Ex.*** Find the ***critical numbers*** of the function.  f(x) = 2x3 + 3x2 – 36x  ***Solution.***  f’(x) = 6x2 + 6x – 36  f’(x) = 0 x = 2, x= - 3  critical numbers: 2 and -3. | **24/** Find the ***critical numbers*** of the function  f(x) = f(x) = x4 - 2x2 + 3. |
| **increasing/decreasing**  **local (relative) min/max**: 1st derivative test and 2nd derivative test  **concave upward/downward**  **inflection points** | ***Ex1.*** The graph of the derivative of a function is shown.    a/ On what intervals is *f* increasing or decreasing?  b/ At what values of x does *f* have a local maximum or minimum?  ***Solution.***  a/ Based on the graph above, f’(x) < 0 on the intervals (0, 1) and (5, 6) 🡺 f is decreasing on (0, 1) and (5, 6); f is increasing on (1, 5) because f’(x) > 0 on (1, 5).  b/ f’ changes sign from (-) to (+) at x = 1 🡺 f has local minimum at x = 1.  f’ changes sign from (+) to (-) at x = 5 🡺 f has local maximum at x = 5.  ***Ex2.*** Given f(x) = 2x3 + 3x2 – 36x  a/ Find the intervals on which is *f* increasing or decreasing.  b/ Find the local maximum and minimum values of *f* .  c/ On what intervals is *f* concave upward or concave downward?  d/ Find all inflection points of *f*.  ***Solution.***   * f’(x) = 6x2 + 6x – 36   f’(x) = 0 x = 2, x= - 3  sign of f’  x -∞ -3 2 ∞  f’ + 0 - 0 +  a/ f is increasing on (-∞, -3), and increasing on (2, ∞)  f is decreasing on (-3, 2).  b/ relative max: f(-3) = 81, relative min: f(2) = -44   * f’’(x) = 12x + 6   f’’(x) = 0 x = - ½  sign of f’’  x -∞ - ½ ∞  f’’ - 0 +  c/ f is concave downward on (-∞, - ½) and concave upward on (- ½, ∞)  d/ at x = - ½, f changes from concave downward to concave upward 🡺 inflection point is (-1/2, f(-1/2))  or (-1/2, 20) | **25/** The graph of the ***derivative*** of a function is shown.  a/ On what intervals is *f* ***increasing*** or ***decreasing***?  b/ At what values of x does *f* have a relative ***maximum or minimum***?    **26/** Given f(x) = x4 - 2x2 + 3  a/ Find the intervals on which is *f* increasing or decreasing.  b/ Find the ***relative maximum*** and minimum values of *f* .  c/ On what intervals is *f* concave upward or concave downward?  d/ Find all ***inflection points*** of *f*. |
| **abs. max/min** and  **Optimization problems** | ***Ex1.*** Find two numbers whose difference is 40 and product is minimum.  ***Solution.***  We find x and y such that x – y = 40 and x.y is minimum.  Let f(x) = x.y = x.(x-40) = x2 – 40x  f’(x) = 2x – 40  f’(x) = 0 x = 20  and f’’(20) = 2 > 0   * f(20) = -400 is minimum value of f.   So, x = 20 and y = -20  ***Ex2.*** A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit, in dollars, must be p(x)= 400 - x. The manufacturer also determines that the total cost of producing x units is given by C(x) = 1000 + 40x.  How many units must the company produce and sell in order to ***maximize profit***?  ***Solution.***  Profit P(x) = R(x) – C(x) = x.p(x) – C(x)   * P(x) = x(400 – x) – (1000 + 40x) * P(x) = -x2 + 360x – 1000   All we want is to find x so that P(x) is maximum.  P’(x) = -2x + 360  P’(x) = 0 x = 180  P’’(x) = -2 < 0 🡺 P’’(180) < 0  By 2rd derivative test, P(180) is the maximum value of P. | **27/** Find two numbers whose difference is 40 and product is ***minimum***.  **28/** Find the absolute ***maximum*** and ***minimum*** of the function f(x) = x3 – 2x2 + 5x – 1 on [0, 3].  **29/** Find the ***maximum*** value of f(x) = x3(1-x)4, 0 ≤ x ≤1.  **30/** Sound software estimates that it will sell N units of a program after spending a dollars on advertising, where N(a) = - a2 + 100a + 2, 0 ≤ a ≤ 100, and a is in thousands of dollars. Find the ***maximum*** number of units that can be sold.  **31/** Let P(x) = -2x2+120x + 27 be the profit function.  Find the ***maximum profit*** and the number of units, x, that must be produced and sold in order to yield the ***maximum profit***. |
| Relative extrema | ***Ex.*** Find the ***relative extrema*** of the function f(x) =  ***Solution.***  f’(x) =  f’(x) = 0 when x = 0.  From f’(x) = , we can see f’(x) changes sign from – to + at x = 0.  So, f(0) is the ***relative minimum*** of f. | **32/** Find the ***relative extrema*** of the function |
| **Chapter 4: Integration** | | |
| **Integrals and**  **areas,**  **Riemann sum** | ***Ex.*** Given f(x) = 6x2 – 4x  a/ Approximate the area under f(x) from x = 1 to x = 4 using Riemann sum with n = 6 and ***left endpoints***.  b/ Find the area under f(x) from x =1 to x = 4 by computing the integral dx.  ***Solution.***  a/   * [1, 4] is divided into 6 subintervals: [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3], [3, 3.5], [3.5, 4] * Left endpoints: 1, 1.5, 2, 2.5, 3, 3.5 * Area ≈ (f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5)) = 77.25000000     b/Actual area = dx = dx = 96. | **33/** Given f(x) = 3x2 – 2x  a/ Approximate the dx by computing the ***area*** under f(x) using **Riemann sum** with n = 4 and ***right endpoints***.  b/ Find the ***area*** under f(x) from x = 0 to x = 8. |
| **= F(b) – F(a)** | ***Ex1.*** Given f(1) = 3, f’ is continuous and dx = 7. Find f(4).  ***Solution.***  f is an antiderivative of f’ 🡺 dx = f(4) – f(1)   * f(4) – f(1) = 7 🡺 f(4) = 7 + 3 = 10   ***Ex2.*** Suppose h is a function such that  h(1) = -2, h’(1) = 2, h’’(1) = 3, h(2) = 6, h’(2) = 5, h’’(2) = 13 and h’’ is continuous everywhere.  Evaluate dx.  ***Solution.***  h’ is an antiderivative of h’’ 🡺  dx = h’(2) – h’(1) = 5 – 2 = 3. | **34/** Given dx =  Find b.  **35/** Compute dx, where  Hint:  dx = dx + dx |
| **Average value** of f(x) over [a, b] | ***Ex.*** Find the ***average value*** of the function f(x) = 3x2 – 2xover [1, 3].  ***Solution.***  fave = = = 18/2 = 9 | **36/** Find the ***average profit*** when the company sells first 50 products. Given R(x) = 2000 + 4x and C(x) = 12x3+ 1000x.  **37/** Suppose the ***average value*** of f over [1, 5] is 7/2.  Find |
| udv = uv - vdu  (Integration by parts) | ***Ex1.*** Find 4xe-2xdx  ***Solution.***  Let u = 4x, dv = e-2xdx 🡺 du = 4dx, v =  So, 4xe-2xdx = udv = uv - vdu  = -2xe-2x + 2e-2xdx = -2xe-2x - e-2x + C  ***E2.*** Find dx  ***Solution.***  Let u = lnx, dv = 2xdx 🡺 du = dx/x, v = x2  dx = = uv -  = x2lnx - = x2lnx - x2  = e2 - (e2 – 1) = e2 + ½ | **38/** Find the integrals:  a/ 2xe-xdx  b/ 4xln(2x)dx  Hint:  u = ln(2x), dv = 4xdx   * du = u’dx   = dx = dx/x  c/ dx  Hint:  u = 2x, dv = dx   * v = 2 |
| f(x)dx = g(t)dt  by substitution rule with t = u(x) | ***Ex1.*** Find dx  ***Solution.***  Let t = 🡺 dt = 2xdx  So, dx = t9dt = + C  = + C  ***Ex2.*** Find dx  ***Solution.***  Let t = lnx 🡺 dt = dx/x  So, dx = t dt = t2 + C = (lnx)2 + C | **39/** Evaluate the integrals:  a/  (Hint: t = x3 + 1)  b/  (Hint: t = )  c/  (Hint: t = ln(x)) |
| **Chapter 5. Applications of integration** | | |
| **improper integral** | ***Ex1.*** Evaluate the improper integral  ***Solution.***  = -2 = -2( e0) = -2(0 – 1) = 2  ***Ex2.***   * If p > 1,  🡺 the integral is convergent. * If p ≤ 1, the integral diverges to ∞. | **40/**  **41/**  **42/** |
| Total cost  Total profit  Total revenue | ***Ex1.*** A company determines that the ***marginal cost***, C’, of producing the xth unit of a product is given by C’(x) = 12x3 - 4x. Find the ***total cost*** function, C(x), assuming that C(x) is in dollars and that fixed costs are $1200.  ***Solution.***  Total cost C(x) = C’(x)dx  C(x) = (12x3 – 4x)dx = 3x4 – 2x2 + D  fixed costs are $1200 🡺 C(0) = 1200   * D = 1200. * Total cost function: C(x) = 3x4 – 2x2 + 1200 (USD)   ***Ex2.*** Assume that R’(x) = 3x2 – 2 is the marginal revenue, in dollars, from selling xth unit of a product. Find the ***total revenue*** function R(x), if we know R(0) = 0.  ***Solution.***  Total revenue function R(x) = R’(x)dx  R(x) = (3x2 – 2)dx = x3 – 2x + D  R(0) = 0 🡺 D = 0.  So, total revenue function R(x) = x3 – 2x. | **43/** The cost per yard of producing x yards of a particular fabric is given by C’(x) = 0.003x + 12, for x ≤ 550, where C’(x) is the cost in dollars. Find the ***total cost*** of producing 200 yd of this material.  **44/** An air conditioning company determines that the marginal cost, in dollars, for the xth air conditioner is given by C’(x) = -0.4x + 100,  C(0) = 0. Find the ***total cost*** of producing 45 air conditioners. |
| Position, velocity, acceleration  Distance traveled | ***Ex.*** A particle starts out from the origin. Its ***velocity***, in m per minute, after t minutes is given by v(t) = 3t2 - 12t. ***How far*** does it ***travel*** after 5 minutes?  ***Solution.***  v(t) = 3t2 - 12t = 3t(t – 4)  Sign of v(t):    Distance traveled =    = … (sorry, I have no any calculator) | **45/** A particle starts out from the origin. Its ***velocity***, in miles per hour, after t hours is given by v(t) = 12t3 + 4t. ***How far*** does it ***travel*** from the start through the 2rd hour? |
| Demand function D(x),  Supply function S(x)  Equilibrium point  Consumer surplus,  producer surplus | ***Ex1.*** Given D(x) = (x – 5)2 and S(x) = x2 + x +3, in which D(x) is the demand function and S(x) is the supply function. Find ***equilibrium point*** and the ***consumer surplus*** at equilibrium point.  ***Solution.***  D(x) = S(x) (x – 5)2 = x2 + x +3  x2 – 10x + 25 = x2 + x +3  x = 2   * D(2) = S(2) = 9   Equilibrium point: (2, 9).   * Consumer surplus     = …  ***Ex2.*** Let D(x) is the price, in dollars per unit, that consumers are willing to pay for x units of an item;  S(x) is the price, in dollars per unit, that producers are willing to accept for x units.  Given D(x) = x2 -4x + 7 and S(x) = x2 + 8x - 29, find the ***consumer surplus*** and the ***producer surplus*** at the equilibrium point.  ***Solution.***   * D(x) = S(x)   x2 -4x + 7 = x2 + 8x – 29  x = 3  D(3) = S(3) = 4   * Equilibrium point: (3, 4) * Consumer surplus:     = …   * Producer surplus:     = … | **46/** Find the ***producer surplus*** for S(x) = 2x2 - 3x + 7 when x = 2.  **47/** Given D(x) = -x + 6 and S(x) = x2 -3x + 3, find the ***consumer surplus*** at the ***equilibrium point***.  Answer: **(3, 3) and4.5**  **48/** Given D(x) = -x2 +7x + 3 and S(x) = 3x - 2, find the ***equilibrium point*** and the ***producer surplus*** at this point.  Answer: (5,13) and 37.5  **49/** Find the ***consumer surplus*** for the demand function given by D(x) = -3x2 + 6x + 3 when x = 5.  **50/** Let D(x) is the price, in dollars per unit, that consumers are willing to pay for x units of an item;  S(x) is the price, in dollars per unit, that producers are willing to accept for x units.  If D(X) = -2x + 12, S(x) = 2x -4, find the ***consumer surplus*** at the ***equilibrium point.*** |
| Future value  present value  Accumulated  future value from money flow  accumulated present value from money flow | ***Theory.***   * P0 : initial deposit invested * k: compound interest rate per year, continuously * P = P0ekt : future value after t year. * P0 = Pe-kt: present value * R(t): money flow (yearly) * A =  : accumulated future value after T years * B = : accumulated present value * : accumulated present value for a ***perpetual*** continuous money flow.   ***Ex1.*** Find the ***future value*** P of the amount P0 = 55000 dollars invested for time t = 12 yr and k = 6%, compounded continuously.  ***Solution.***  P = P0ekt = 55000.e0.06\*12 ≈ 112993.8 (USD).  ***Ex2.*** In 18 yr, Claire Beasley is to receive 120000 dollars under the terms of a trust established by her aunt. Assuming an interest rate of 6.5%, compounded continuously, what is the ***present value*** of Claire’s legacy?  ***Solution.***  P = 120000, t = 18, k = 0.065  Present value = P0 = Pe-kt = 120000\*e-0.065\*18 ≈ 37244.06 (USD).  ***Ex3.*** At age 25, Del earns his CPA and accepts  a position in an accounting firm. Del plans to retire at  the age of 65, having received an ***annual salary*** of  $125,000. Assume an interest rate of 7%, compounded continuously.  a) What is the ***accumulated present*** value of his position?  b) What is the ***accumulated future value*** of his position?  ***Solution.***   1. R(t) = 125,000   k = 0.07  T = (65 - 25) = 40  Accumulated present value:  B =   1. R(t) = 125,000   k = 0.07  T = (65 - 25) = 40  Accumulated future value:  A= | **51/** Upon the death of his uncle, David receives an inheritance of $50,000, which he invests for 16 yr at 7.3%, compounded continuously. What is the future value of the inheritance?  **52/** Upon the death of his aunt, Burt receives an inheritance of $80,000, which he invests for 20 yr at 8.2%, compounded continuously. What is the future value of the inheritance?  **53/** At age 35, Rochelle earns her MBA and accepts a position as vice president of an asphalt company. Assume that she will retire at the age of 65, having received an annual salary of $95,000, and that the interest rate is 6%, compounded continuously.  a/ What is the accumulated present value of her position?  Answer:  b/ What is the accumulated future value of her position?  Answer: |
| Differential equations | ***Ex1.*** Solve for y if y’ = -3x2 + 4x – 5, and y = 3 with x = 1.  ***Solution.***   * y’ = -3x2 + 4x – 5 🡺 y = -x3 + 2x2 – 5x + C * When x = 1, y = 3 🡺 C = 7 * So, y = -x3 + 2x2 – 5x + 7.   ***Ex2.*** Solve the equation y’ = x.y2, y = 2 when x = 1.  ***Solution.***   * Note that y’ = dy/dx * dy/dx = x.y2 🡺      * When x = 1, y = 2 🡺 * From , it follows that     ***Ex3.*** Solve the differential equation: y’ = 2x - 2xy; y = 3 when x = 0.  ***Solution.***   * y’ = dy/dx * dy/dx = 2x – 2xy = x(1 – y)     where D = , an unknown constant.   * y = 3 when x = 0 🡺 3 = 1 – D.e-0 🡺 D = -2. * So, . | **54/** Solve for y if y’ = e-2x + 2x – 1, and y = 2 when x = 0.  Answer: y =  **55/** Solve the differential equation:  Answer: y + siny = x3 + C  **56/** Solve the differential equation: y’ = 6x2y; and y = 2 if x = 0.  **57/** Solve the differential equation y.y’ = x/2, and y = 3 if x = 0.  **58/** Solve the differential equation: y’ = 2x + 2xy; y = 2 when x = 0.  **59/** Solve the differential equation y’y = 1; y = 3 when x = 1.  **60/** Solve the differential equation y’ = 3 - y; y = 2 when x = 0.  **61/** Stock growth. The growth rate of a certain stock, in dollars, can be modeled by  where V is the value of the stock, per share, after months; k is a constant; L = $24.81, the limiting value of the stock; and V(0) = 50. Find the solution of the differential equation in terms of t and k. |
| **Chapter 6. Functions of Several Variables** | | |
| Domain of f(x,y)  Find f(x,y) | ***Ex1.*** For f(x,y) = log10(x2 + y2) + 2x – y, find f(1, 0), f(3, 1) and f(6, 8).  Solution.   * f(1,0) = log10(12 + 02) = log10(1) = log10(100)= 0 * f(3,1) = log10(32 + 12) = log10(10) = log10(101) = 1 * f(6,8) = log10(62 + 82) = log10(100) = log10(102) = 2   ***Ex2.*** Find the domain of the function  ***Solution.***     * Domain: the set of points   ***Ex3.*** The total cost to a company, in thousands of  dollars, of producing its goods is given by  C(x, y, z, w) = 4x2 + 5y + z – ln(w + 1), where dollars are spent for labor, dollars for raw materials, dollars for advertising, and dollars for machinery. This is a function of four variables (all in thousands of dollars). Find C(3, 2, 0, 10).  ***Solution.***  C(3, 2, 0, 10) = 4·32 + 5·2 + 0 – ln(10 + 1) ≈ 43.6 thousands of dollars. | **62/** The price–earnings ratio (PE Ratio) of a stock  is given by  , where P is the price of the stock and E is the earnings per share (EPS). The price per share of Apple Inc. stock was 179.97 USD and the earnings per share were 9.70 USD. Find the price–earnings ratio.  Answer: 18.55  **63/** A hockey goaltender’s goals  against average A is a function of the number of goals g  allowed and the number m of minutes played and is  given by the formula  a/ Determine the goals against average of a goaltender who allows 35 goals while playing 820 min. Round A to the nearest hundredth.  b/ A goaltender gave up 124 goals during the season and had a goals against average of 3.75. How many minutes did he play? (Round to the nearest integer.)  **64/** Determine the ***domain*** of the functions:  a/  b/ |
| Partial derivatives  fx: partial derivative of f with respect to x  (the same notation for partial derivatives)  (eu)’ = u’·eu  (ax)’ = ax·lna | ***Ex1.*** For f(x, y) = 3x2y + xy, find and fx,  and evaluate fx at (2, -3).  ***Solution.***   * fx = 6xy + y (Treating y as a constant) * = 3x2 + x (Treating x as a constant) * fx(2,-3) = 6(2)(-3) + (-3) = -39.   ***Ex2.*** For g(x, y) = exy + ylnx, find gx, gy and evaluate gy at (1, -1).  ***Solution.***     * = 1/e.   ***Ex3.*** For , find kx, ky and ky(3,4).  ***Solution.***          * .   ***Ex4.*** For f(x, y) = x2y3 + x4y + xey, find fxx, fxy, fyx, fyy.  Solution.  fx = 2xy3 + 4x3y + ey  fy = 3x2y2 + x4 + xey  fxx =  fxy =  fyx =  fyy = . | **65/** For f(x, y) = e2x-y, find fx, fy and fx(1, 1).  **66/** For f(x, y) = , find fx, fy and fy(3,1).  **67/**  a/ For , find kx, ky and ky(2,4).  b/ For z = x2 – 2x/y, find zx(2, 1) – zy(1, 2)  **68/** Find and gx, gy and gλ if  g(x, y, λ) = x2 + y2 - λ(10x + 2y – 4).  **69/** For f(x, y) = 6x2 + 3xy4 – y2, find fxx, fxy, fyx, fyy.  **70/**For T(x, y) = x2 + 2y2 -8x + 4y, find (x, y) such that .  **71/** For  find (x, y) such that . |
| Cobb–Douglas production function p(x, y)  marginal productivity of labor px  marginal productivity  of capital py  Higher partial derivatives  Note: often (but not always), fxy = fyx). | ***Ex1.*** A cellular phone company has the following production function for a smart phone:  p(x, y) = 50x2/3y1/3, where p is the number of units produced with x units of labor and y units of capital.  a/ Find the number of units produced with 125 units of labor and 64 units of capital.  b/ Find the ***marginal productivities***.  c/ Evaluate the ***marginal productivities*** at and y = 64.  ***Solution.***  a/ p(125, 64) = 50(125)2/3(64)1/3 =  b/ marginal productivity of labor  marginal productivity of capital  c/ For 125 units of labor and 64 units of capital, we have:   * marginal productivity of labor * marginal productivity of capital | **72/** A publisher’s production  function for textbooks is  given by  p(x,y) = 72x0.8y0.2, where p is the number of books produced, x is units of labor, and y is units of capital. Determine the ***marginal productivities*** at x = 90 and y = 50. |
| relative extrema | The D-Test   1. Find fx, fy, fxx, fyy, fxy 2. Find (x, y) such that fx = 0 and fy = 0. Let (a, b) be a solution. 3. Evaluate D = fxx(a, b).fyy(a, b) – [fxy(a, b)]2 4. Then, 5. If D < 0 🡺 f has a *saddle point* at (a, b) 6. If D > 0 🡺 f has *relative extrema* at (a, b) and:  * If fxx(a, ) < 0 🡺 maxima * If fxx(a, b) > 0 🡺 minima  1. D = 0 🡺 the test is NA (not applicable)   ***Ex1.*** Find the ***relative maximum*** and ***minimum*** values and the ***saddle points***.  a/ f(x, y) = x2 + xy + y2 -3x  b/ f(x, y) = 6xy – x3 –y2  ***Solution.***  a/ f(x, y) = x2 + xy + y2 -3x   * fx = 2x + y – 3   fy = x + 2y  fxx = 2  fyy = 2  fxy = 1   * Solve the system fx =0 and fy = 0      * D = fxx(2, -1)·fyy(2, -1) – [fxy(2, -1)]2 = 2·2 – 12 = 3 > 0   and fxx = 2 > 0  Conclusion: D > 0 and fxx > 0 🡺f has ***relative minimum*** at (2, -1) and f(2, -1) = -9.  b/ f(x, y) = 6xy – x3 –y2   * fx = 6y – 3x2   fy = 6x – 2y  fxx = -6x  fyy = -2  fxy = 6   * Solve the system fx =0 and fy = 0     From (2), y = 3x  So (1) becomes 18x – 3x2 = 0  3x(6 – x) = 0  x = 0 or x = 6   * x = 0 🡺 y = 0 * x = 6 🡺 y = 18 * D(0,0) = fxx(0, 0)·fyy(0, 0) – [fxy(0, 0)]2 = 0·(-2) – 62 = -36 < 0 * (0, 0, 0) is a ***saddle point***. * D(3,18) = fxx(3, 18)·fyy(3, 18) – [fxy(3, 18)]2 = -18(-2) – 62 = -72 < 0. * (3, 18, -27) is a ***saddle point***.   ***Ex2.*** **Maximizing Profit.** A firm produces two kinds of golf ball, one that sells for $3 and one priced at $2. The total revenue, in thousands of dollars, from the sale of x thousand balls at $3 each and y thousand at $2 each is given by  R(x, y) = 3x + 2y  The company determines that the total cost, in thousands of dollars, of producing x thousands of the $3 ball and y thousands of the $2 ball is given by  C(x, y) = 2x2 - 2xy + y2 -9x + 6y + 7  How many balls of each type must be produced and sold in order to ***maximize profit***?  Solution.  Profit function P(x,y) = R(x,y) – C(x,y)  = -2x2 – y2 +2xy +12x -4y -7  We want to ***maximize*** P(x,y).   * Px = -4x + 2y + 12   Py = -2y + 2x – 4  Pxx = -4  Pxy = 2  Pyy = -2   * Px = 0 and Py = 0   -4x + 2y + 12 = 0 and -2y + 2x – 4 = 0  x = 4 and y = 2   * D(4, 2) = Pxx(4,2)·Pyy(4,2) – [Pxy(4,2)]2 = -4(-2) – 22 = 4 > 0   Pxx(4,2) = -4 < 0 and D(4,2) > 0 🡺 P has maximum value at (4, 2).  So, the maximum value is P(4, 2) = $13 thousands. | **73/** Find the relative maximum and minimum values and the *saddle points*.  a/ f(x, y) = x2 + y2 – 2x + 4y -2  b/ f(x, y) = 4xy – x2 –y2  **74/** A flat metal plate is located on a coordinate plane. The temperature of the plate, in degrees Fahrenheit, at point is given by  T(x, y) = x2 + 2y2 -8x + 4y Find the ***minimum*** temperature and where it occurs.  **75/** The Zshop buys two kind of clothes, jean and T-shirt. The total cost to produce x units of jean and y units of T-shirt, in thousand is given by C(x,y) = x2+ y2 - 8x-4y+40 (in thousand dollar).  Find the ***minimum cost***? |
| Maximum and minimum with **constrains**  The Method of Lagrange Multipliers | ***Method.*** To find a maximum or minimum value of a function f(x, y) subject to the constraint g(x, y) = 0:   * **Step 1:** Form a new function,   F(x, y, λ) = f(x, y) - λg(x, y)   * **Step 2:** Find   Fx, Fy, Fλ   * **Step 3:** Solve the system   🡺 solution(a, b, λ).   * **Step 4:** Evaluate the maximum/minimum value at (a, b).   ***Ex1.*** Find the maximum value of A(x, y) = xy subject to the constrain x + y = 20.  ***Solution.***   * x + y = 20 🡺 x + y – 20 = 0, so g(x, y) = x + y – 20   Lagrange function F(x, y) = xy - λ(x + y – 20)   * Fx = y - λ   Fy = x - λ  Fλ = -(x + y – 20)   * Solve the system: Fx = 0, Fy = 0, Fλ = 0        * The maximum value of subject to the constraint occurs at (10, 10) and is A(10, 10) = 100.   ***Ex2.*** Find the minimum value of f subject to the given constraint f(x,y)= x2 + y2; 4x + y = 20.  ***Solution.***   * 4x + y = 20 🡺 4x + y – 20 = 0 * g(x, y) = 4x + y -20.   Lagrange function F(x, y) = x2 + y2 - λ(4x + y – 20)   * Fx = 2x - 4λ   Fy = 2y - λ  Fλ = -(4x + y – 20)   * Solve the system: Fx = 0, Fy = 0, Fλ = 0        * The ***minimum value*** of subject to the constraint occurs at (80/17, 20/17) and is f(80/17, 20/17) ≈ 23.53. | **76/** Find the ***maximum value*** of f subject to the given constraint.  a/ f(x, y) = xy; 3x + y = 10  b/ f(x, y) = 4 – x2 – y2; x + 2y = 10  **77/** Find the ***minimum value*** of f subject to the given constraint.  a/ f(x, y) = x2 + y2; 2x + y = 10  b/ f(x, y) = 2y2 – 6x2; 2x + y = 4.  **78/** **Maximizing total sales.** The total sales, S, of a firm are given by  S(M, L) = ML – L2, where M is the cost of materials and L is the cost of labor.  Find the ***maximum value*** of this function subject to the budget constraint  M + L = 70. |
| The **least square** method  Linear **REGRESSION**  **Regression line** | **Theory.**  Regression is a technique for determining a continuous function that “**best fits**” a set of data points.  The ***Regression Line*** for a Collection of Data Points (c1, d1), (c2, d2),…, (cn, dn)    Use a **calculator** to find the REGRESSION LINE for the data points (c1, d1), (c2, d2),…, (cn, dn):  y = mx + b  With a typical calculator ***CASIO FX 570ES PLUS***    ***Ex1.*** Find the ***regression line*** for the data points (1, 5.2), (2, 8.9), (3, 11.7), and (4, 16.8).  ***Solution.***  The ***regression line*** is of the form y = mx + b  (Use a calculator:  b =1.25  m = 3.76  y = 3.76x + 1.25  ***Ex2.*** **Labor force.** The minimum hourly wage in the United States has grown over the years, as shown in the table below.    a/ For the data in the table, find the regression line, y = mx + b.  b/ Use the regression line to ***predict*** the minimum hourly wage in 2015 and 2020.  Solution.  a/ (Use a calculator): y = 0.1468x + 3.9452  b/ In 2015: x = (2015 – 1990) = 25 🡺 y = 0.1468(25) + 3.9452 = 7.6152.  In 2020: x = (2020 – 1990) = 30 🡺 y = 0.1468(30) + 3.9452 = 8.3492. | **79/** Find the ***regression line*** for the given data   |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | x | 1 | 2 | 3 | 4 | 5 | | y | 3 | 4 | 4 | 6 | 8 |   **80/** Consider the data in the following table regarding enrollment in colleges and universities during a recent 3-year period.    a/ Find the ***regression line***, y = mx + b.  b/ Use the ***regression line*** to predict enrollment in the  fourth year.  **81/** Consider the data in the table below regarding workers’ average monthly out-of-pocket premium for health insurance for a family.    a/ Find the regression line, y = mx + b.  b/Use the regression line to predict workers’ average monthly out-of-pocket premium for health insurance for a family in 2012. |

**THE END**